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A Rayleigh–Ritz based approach to characterize the vertical vibration of non-uniform hull girder

N. Datta [a,*], J.D. Thekinen [b]

[a] Indian Institute of Technology, Kharagpur, India
[b] Department of Mechanical Engineering, Purdue University, West Lafayette, USA

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ABSTRACT

Previous attempts to solve for vibration of non-uniform hull girder have used Finite Element Analysis (FEA). This work utilizes the Rayleigh–Ritz (R–R) method to analyze the vertical vibration of a non-prismatic mathematical hull, with arbitrary longitudinal distribution of sectional properties. It is shown that (a) the R-R method provides reasonably accurate results for the actual natural frequency; and (b) the R–R method offers significant computational advantages over FEA. This computational supremacy can be exploited in the initial design stages when several designs are to be iteratively tested for its structural characteristics. The natural frequencies and modeshapes are obtained by the Rayleigh–Ritz method. The non-uniform beam modeshape is a weighted series sum of the (closed-form) uniform beam modeshapes. Only a few uniform beam modes are necessary to generate the non-uniform beam modeshape and a convergent hull girder frequency. Thus, the method is very effective in the low-frequency domain like ship-structure. The proposed method is compared with FEA for two illustrative structures : (a) a containership and (b) a tanker; and also with several hulls from established literature. Modal convergence studies have also been included. The distortions of the non-uniform modeshapes have been studied in the light of the loading conditions of the hull.

1. Introduction

Investigation of ship hull girder vibration is very important at the initial stages of ship design, especially with large ocean going vessels, having longer and larger dimensions. Ship design must be judged in the light of hydroelasticity for the structural integrity and performance of the vessel. Long ships with shallow depth and draft are comparatively flexible in longitudinal bending and become prone to 2-noded vertical hull girder vibration (first noticed in 1967).

Ship springing is the resonant response of the ship to hydrodynamic excitation caused by incident waves. It is a low-frequency, large amplitude, steady-state phenomenon. Springing is a continuous process excited by waves at an encounter frequency equal to the fundamental wet natural frequency of the hull. It becomes more important at increased speeds, lengths, and hull girder flexibilities. The resonant frequency occurs at the high-frequency tail of the sea spectrum. The total deflection is a superposition of the dry hull flexural modes. Springing is the most pronounced in head seas, in lighter sea states, when the T_z (average zero-up crossing period) is small. Full-form tankers are long enough to be modeled as beams. This is possible for slender ships with large L/D ratio (for vertical bending), and large L/B ratios (for horizontal bending). Beam theories are used at the initial design stages, which are good for the first few modes of vibration (Bishop and Price, 1979). Usually, the hull is considered as a free-free Timoshenko beam with varying sectional properties. We need an accurate weight and bending stiffness distribution to calculate the dry natural frequencies of the hull.

The objective of this work is as follows :

- Application of the Rayleigh–Ritz method to an Euler-Bernoulli beam with arbitrarily varying mass and stiffness distributions. A higher-order non-uniformity in section area of beam can be handed easily as the methodology involves integral equations.
- Demonstrating the efficacy of the Rayleigh–Ritz method as an alternative to FEA in the vibration analysis of a hull girder in the early stages of ship design.
- Using closed-form admissible functions of a free-free Euler–Bernoulli beam, which are the uniform beam modeshapes, in the above method for analysing a hull girder for its dry vertical vibration frequencies.
- Generating converged non-uniform modeshapes for the hull

* Correspondence to: Department of Ocean Engineering and Naval Architecture, Indian Institute of Technology, Kharagpur, India.
E-mail address: nabanitadatta@gmail.com (N. Datta).

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Nomenclature

L	Length of vessel
B	Breadth
D	Depth of vessel
x	Independent space variable along the ship length
ρ_w	Density of water
$m(x)$	Mass per unit length
ρ	Density of the material
E	Modulus of elasticity of the material
$I(x)$	Second moment of area for the cross section about horizontal neutral axis
$z(x, t)$	Vertical vibratory displacement
$\Phi_j(x)$	jth non-uniform beam flexural mode
$\varphi_k(x)$	kth uniform beam modeshape with free end condition
γ_j	Frequency parameter of the jth uniform beam modeshape
a_{jk}	Weight of contribution of jth uniform beam modeshape to kth non-uniform beam modeshape
ω_j	jth natural frequency of vibration
ω_{jND}	Non-dimensionalized jth natural frequency
I_0	Vertical 2nd moment of the midship section
A_0	Structural cross sectional area of the midship section
β_{jk}	Element (j,k) of the generalized mass matrix
α_{jk}	Element (j,k) of the generalized stiffness matrix
λ	Eigen values of the Rayleigh Ritz method
T	Total kinetic energy of the beam
U	Total potential energy of the beam

girder from the above method, bypassing the cumbersome and expensive experimental techniques (Zhu et al., 2011). They can be used in the normal-mode summation-based hydroelastic analysis of wave-induced flexural responses, in the design spiral.

Timoshenko (1937) applied the theory of bar vibration with variable cross-section to hull girder vibration, using the Rayleigh–Ritz method. A mathematical hull, with fore-aft symmetry and parabolic mass and stiffness distributions, was analyzed of the first two natural frequencies, using the first two 'symmetric' uniform free-free beam modeshapes. Bishop and Price (1979), in their pioneering work on hydroelasticity, considered hull girder to be both Euler–Bernoulli and Timoshenko beams for the analysis, with the mass and stiffness distributions varying arbitrarily along the length. However, the explicit expression for the non-uniform beam modeshapes was not used in the subsequent "in-vacuo" free vibration analysis. Two practical hulls were analyzed for their vertical frequencies using uniform beam theory and the Prohl–Myklestad method.

1.1. Non-uniform beam vibration

Enough literature is available on the vibration analysis of non-uniform beams, using energy-based variational methods and FEA; dealing with beams whose non-uniformity varied mathematically along the length. Abrate (Qiusheng et al., 1996) studied Euler–Bernoulli beams with parabolically varying thickness, applying the Rayleigh–Ritz method using polynomial admissible functions, to calculate the natural frequencies. Qisheng et al. (1996) used polynomial/exponential mass and stiffness distributions, and Bessel functions in the self-conjugate GDE to establish the non-uniform modes and corresponding natural frequencies. Laura et al. (1996) studied beams with bilinearly varying cross-sections using optimized the Rayleigh–Ritz method and FEA, whose results were in good agreement. Zhou and Cheung (2000) studied linearly and bi-linearly tapered beams with the Rayleigh–Ritz method, but static deflections as the admissible functions in the process. For Auciello (2001), the admissible functions were linearly independent orthogonal polynomials generated by the iterative Gram–Schmidt orthonormalization; to study an axially loaded linearly tapered beam. Ece et al. (2007) considered the beam with exponentially varying thickness.

From this literature, we come to know the following:

- The variational nature of the R–R method leads to an upper bound of frequencies, and to a monotone convergence as the number of admissible functions goes to infinity.
- The mathematical closed-form variation of the cross-section gives closed-form solutions of the non-uniform modeshapes, and thus, very accurate frequencies.
- The frequency ratio (ratio of non-uniform beam frequency to the uniform beam frequency) was seen to be smaller for higher mode numbers. Thus, the effect of non-uniformity is more pronounced for the lower modes.

However, for a typical merchant vessel, the hull girder has arbitrary (non-mathematical) distributed mass and stiffness distributions, often showing saw-tooth-like variations, as shown in literature (Bishop and Price, 1979; Senjanovic et al., 2009; Seng et al., 2012; Zhou and Zhao, 2006; Hirdaris et al., 2003). The non-uniformity influences the natural frequencies and modeshapes. The closed-form expression of hull girder flexural modes cannot be generated through any of the above analytical methods. Closed-form solutions are more attractive to the structural dynamics analyst, to further use them in the hydroelastic response analysis of the vessel.

1.2. Ship hull vibration analysis

Till the 1960's, ship designers used empirical formulations, treating the ship as a free-free beam, to calculate the natural hull girder vibration frequencies in the early stages of design. The empirical relations were based on past data, as shown by Todd (1961) and Kumai (1967) and the SR94 Panel. Classification society rules do not explicitly account for the effect of hull girder flexibility on the global loads. From 1970s, when ships became longer and faster, a more scientific approach was required to characterize the flexural behavior of vessels. Increase in main dimensions makes the vessel more flexible, and the requirement for high speed causes the encounter frequencies of the incident wave spectrum (esp. head seas) to be closer to the wet natural frequencies of the ship's flexible modes. Wet natural frequency $\sim$0.5 Hz for $L > 300$ m. The accurate determination of natural frequencies and modeshapes of the hull is of significant importance for understanding its hydroelastic characteristics. Experiments were conducted by Zhu et al. (2011) with backbone models. However, linear springing is rarely excited in model tests to observe the 2-noded vertical vibration modeshape.

FEA is still the most commonly used vibration analysis technique for ship structures, since the last four decades, as seen in the works of Camiseiti et al. (1980), Skaar and Carlsen (1980), Wan-xic et al. (1983). Hirdaris et al. (2003) used three FEA modules of the commercial FEA code ANSYS to analyze a bulker. Pin and De-you (2006) used dynamic stiffness matrices in a 6-element hull to study a VLCC. Senjanovic et al (2009) analyzed a containership by

FEA for both fully loaded and ballasted conditions, for both vertical and coupled horizontal-torsional vibrations. Pedersen and Jensen (2009) used FEA to analyze the free vibration of a Panamax containership. The 2-noded vibration mode was generated by the least-square approximation, which gave an empirical modeshape for this vessel. Hirdaris et al. (2009) studied the Great Lakes Bulk carrier by modeling it as a 20-element non-uniform Timoshenko beam as explained by Bishop and Price (1979). Along with their experimental efforts, Zhu et al. (2011) relied on FEA to generate the natural frequencies and modeshapes of the backbone model, to be later used for hydroelastic studies. Seng et al. (2012) modeled a post-Panamax container ship as a non-uniform Timoshenko beam, whose non-uniform modes were generated by Stodola's method, which was popular in the 1970s.

1.3. Overview of this work

Here, we explore the scope of Rayleigh–Ritz (RR) approach in identifying the natural frequencies and modeshapes of the non-prismatic hull girder. We provide comparative results with (a) Finite Element Analysis (FEA), (b) empirical formulae and (c) previous literature to show that the accuracy is not compromised in the RR approach. We identify the need to integrate efficient structural analysis in the initial stages of hull design. A vessel is a custom-built product designed based on the owner requirements. No two ships are the same. In the initial design stages, several iterations are performed to optimize the capacity, hydrodynamic drag characteristics and several other variables. The generation of the *hull resonance diagram* requires the hull girder natural frequencies, in all the global modes; and they should not match with the propeller-induced vibration frequency (shaft frequency × number of blades) with an accuracy of ∓5%. The structural dynamics and fluid-structure-interaction characteristics of the final hull form gets analyzed. However, many a times the hull form will be deemed structurally unfit and will have to be re-designed. Thus, integrating structural analysis into the initial design phase is necessary to avoid re-design and reiterating the entire analysis. This can be achieved by improving the efficiency of structural analysis without compromising a lot on solution quality.

Here, we analyze two hull forms of the Indian Merchant Navy, i.e. (a) SCIM Containership and (b) DS Tanker. The dry natural frequencies of vertical hull girder vibration has been analyzed through the energy-based Rayleigh–Ritz method, and verified through Finite Element Analysis, using one-dimensional beam elements of equal length. We conclude new insights about non-uniform beam modeshapes with respect to the spatial correlation of the admissible functions with respect to the mass and stiffness distributions; which can be used to improve the efficiency of RR application. The efficacy of the Rayleigh–Ritz method in calculating the natural frequencies of vertical hull girder vibration is demonstrated by several case studies with the vessels analyzed by Bishop and Price (1979), Hirdaris et al. (2003), and Senjanovic et al. (2009). The frequency-convergence through the R–R method, and the non-uniform modeshapes for these vessels have been established.

The originality of the paper lies in the following :

- Identifying the use of the Rayleigh–Ritz method to analyze the free vertical-plane vibration characteristics of a non-prismatic hull girder in initial design stages, as an alternative to FEA.
- Convergence study of the number of uniform beam modeshapes required, as a weighted sum, in order to generate the dry 'in-vacuo' non-uniform hull girder vibration *frequency* and *modeshape*.
- Analysing several hull forms from published literature (who use empirical methods and/or FEA), and establishing comparable natural frequencies.
- Identifying how the spatial correlation between the property distributions and the uniform beam modeshapes/curvatures influences the distortion of the non-uniform beam modeshapes. The loading condition of the merchant vessel strongly influences the modeshape curvatures and amplitudes.

The structure of the paper is as follows :

- Applying the Rayleigh–Ritz method in analysing the free dry vibration of the hull : generation of the weights of the admissible functions.
- Establishing the converged non-uniform beam natural frequencies, and their corresponding modeshapes.
- Comparison of the above results with FEA studies and empirical formulations of vertical vibration frequency of the hull girder.
- Drawing insights of the distortion of the converged non-uniform beam modeshapes from the corresponding uniform beam modeshape due to their spatial correlations (or lack of it) with respect to the mass and stiffness distributions.

2. Problem formulation and cases studied

The merchant ship hull is modeled as a non-uniform free-free Euler–Bernoulli beam, of length L, with arbitrarily varying mass $m(x)$ (kg/m) and stiffness $EI(x)$ (N m^2) distributions. The distribution does not vary based on a previously defined analytical/mathematical variations, unlike in work (Serge, 1995; Qiusheng et al., 1996; Laura et al., 1996; Zhou and Cheung, 2000; Ece et al., 2007; Timoshenko, 1937; Thomson et al., 1998; Auciello, 2001). The distribution is very arbitrary depending on the design and loading. The hull has been analyzed for dry vertical hull girder vibration, using the energy-based Rayleigh–Ritz method. Finally, comparative results with FEA, empirical estimates, and published work (12 − 16) are presented. The published work includes :

(1) Bishop and Price (1979), Chapter 4 : Destroyer (Euler–Bernoulli beam)
(2) Bishop and Price (1979), Chapter 4 : Tanker (Timoshenko beam)
(3) Hirdaris et al. (2003) : OBO MV Derbyshire Bulker (FEA, equal beam elements)
(4) Senjanovic et al. (2009) : 7800 TEU Containership (FEA, 1D)

2.1. Mathematical hull without fore and aft symmetry (Unlike Timoshenko, 1937)

Table 1 gives the main particulars of the two vessels, provided by the Indian Register of Shipping. Mumbai, India. Once the basic design is obtained, the body plan can be imported into an image reading code in MATLAB which can replace a wide range of body plan sections with equivalent super-ellipses of the form $\left(\frac{y(x,z)}{a(x)}\right)^{p(x)}+\left(\frac{z}{b(x)}\right)^{q(x)}=1$. While local half-breadth and local draught determine the values of $a(x)$ and $b(x)$ respectively, a computationally efficient code is developed which identifies the powers, m and n to define the geometry of the section. For a semi-super-ellipse, with $\theta=\frac{\pi}{2}$, a closed form solution exists for various geometric properties. In the case of ship sections we have $\theta=\frac{\pi}{2}$. Closed-form expressions may be safely employed for evaluation of the geometric properties. The details can be found in Sadowski (2011).

The authors have previously explained in their work Datta and Thekinen (2012) the application of mathematical curves called rectellipses for the generation ship-like sections. Developing on

Table 1
Main particulars of case-study vessels.

Particulars	Unit	SCIM Container	DS Tanker
Length Overall LOA	metre	262	249.98
Length between perpendiculars L	metre	248	239
Moulded Beam B	metre	32.2	44
Moulded Depth D	metre	19.5	21.5
Moulded Draught T	metre	13.2	15.1
L/B	–	7.70	5.43
B/T	–	2.44	2.91
L/D		13.44	11.63
Displacement	Tonnes	74,660	136,011
Light weight	Tonnes	16,875	21,228
Steel weight	Tonnes	15,103	16,747
Semi-concentrated weight	Tonnes	1772.0	4481
Dead weight	Tonnes	57,785	114,783
Block coefficient	–	0.691	0.836
2nd moment of area of cross-section about horizontal neutral axis I_0	m^4	253	610

this idea, the hull form of a containership and tanker was generated. The 3D hull form was generated as a collection of 100 rectelliptic stations. The parameters of the individual rectelliptic stations were varied to closely represent the design of the actual hull-form of the containership and tanker. Assuming a hollow beam with average shell thickness of 16 mm made of steel ($\rho = 7850 kg/m^3$), the longitudinal mass and second moment of area distributions respectively were obtained. These distributions were non-analytic along the length without fore-aft symmetry.

2.2. Non-uniform beam sectional properties distribution

The mass $m(x)$ and stiffness $EI(x)$ distributions are distributed over the Length Overall (LOA) of the vessel, based on the longitudinal distribution of geometric properties. The longitudinal flexural rigidity is provided by : keel plate, side shell, main deck, longitudinal bulkheads, centre girder, side girders, longitudinal frames, deck longitudinals, superstructures, etc. Transverse bulkheads do not participate in longitudinal hull girder stiffness. But they participate in the mass distribution as peaks in regular intervals. The following members of the hull do not contribute to the longitudinal stiffness: transverse bulkheads, floors, margin plates, transverse frames and stiffeners, deck beams, deck transverses, tank side brackets, etc. To obtain the mass distribution $m(x)$, the following procedure was followed:

- Steel weight was calculated from the Hull Numeral by the Watson and Griffin estimation method (Aasen and Bjørhovde, 2010) and was distributed over the length $(0 < x < L)$, based on sectional girth from rectelliptic properties.
- Semi-concentrated part is distributed as (i) engine and main machinery $(0 < x < 0.2L)$; and (ii) deck machinery $(0.9L < x < L)$. The semi-concentrated weights reduce the natural frequency of the hull girder. The farther the semi-concentrated weights are from the nodes of the vibration modeshape, the greater is the associated kinetic energy, the lower are the frequencies of vibration.
- Deadweight was completely distributed between $(0.2L < x < 0.9L)$ based on the sectional area. It adds to the inertia *without* affecting the flexural rigidity of the hull girder. Hence, the loaded ship has a lower natural frequency, closer to the tail-end of the sea spectrum, and thus, is more prone to springing.

3. Analysis methodology

Here, we describe the methodology to study the free vibration of an Euler–Bernoulli beam, with arbitrarily (non-analytically) varying mass and stiffness distributions using the Rayleigh–Ritz approach. The objective is to obtain (i) natural frequencies and (ii) non-uniform modeshapes of the hull. We also mention the other popular approaches of hull girder vibration analysis, to which our results will be compared.

3.1. Non-uniform beam vibration

The steps in this are as follows :

- The admissible functions of the Rayleigh–Ritz (R–R) method are first generated. Here, they are the modeshapes of a uniform (prismatic) free-free beam. They act as orthogonal admissible functions in the R–R method, satisfying the same boundary conditions as the hull.
- The minimization of the natural frequency from an assumed modeshape leads to N number of distinct $\lambda = \omega^2$, i.e. the square of the non-uniform beam natural frequencies.
- Re-instating the $\lambda = \omega^2$ leads to the weights of the admissible functions and hence the non-uniform modeshapes, which influence the potential and kinetic energies of the vibrating non-uniform beam.
- A convergence study gives the number of admissible functions required for the final frequency and modeshape of the non-uniform beam.

3.1.1. Admissible functions

The out-of-plane flexural displacement of the non-uniform beam, undergoing unforced, undamped vibration, in the vertical plane, obeys the following governing differential equation (N/m)

$$m(x)\frac{\partial^2 z(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 z(x,t)}{\partial x^2}\right]=0. \tag{1}$$

There is no axial load on the beam. Pure bending is considered, ignoring shear deformation and rotary inertia. The depth-to-length ratio may be greater than 1/20, but the beam section is mostly hollow, except at the positions of the transverse bulkheads, which do not participate in the longitudinal stiffness. The "thin beam" approximation is assumed here, since the flexural deflections are small. Assuming small-amplitude displacements, where linear superposition holds, the total flexural displacement, in Eq. (1), can be assumed to be a superposition of the modal displacements

$$z(x,t)=\sum_{j=1}^{\infty}\boldsymbol{\Phi}_j(x)q_j(t) \tag{2}$$

In Eq. (2), $\boldsymbol{\Phi}_j(x)$, the non-uniform beam mode, is a weighted sum of the admissible functions, which must satisfy the same boundary conditions. Usually, in the Rayleigh–Ritz method, it is convenient to choose the admissible functions which satisfy the geometric boundary conditions. For a free-free beam, the shear force and bending moment must be zero at the ends (natural boundary conditions). Instead of polynomial admissible functions (Abrate, 1995; Auciello, 2001), or static admissible functions (Zhou and Cheung, 2000), a set of closed-form modes have been used here, as shown below.

The uniform beam modeshape, $\varphi_k(x)$, with free-free end conditions, obeys $\varphi_j''(0)=0, \varphi_j''(L)=0, \varphi_j'''(0)=0, \varphi_j'''(L)=0$. The uniform beam modeshape is given as:

$$\varphi_j(x)=\cos(\gamma_j x)+\cosh(\gamma_j x)+\upsilon_j\left[\sin(\gamma_j x)+\sinh(\gamma_j x)\right];$$
$$\upsilon_j=\frac{\sin\gamma_j L+\sinh\gamma_j L}{\cos\gamma_j L-\cosh\gamma_j L}=\frac{-\cos\gamma_j L+\cosh\gamma_j L}{\sin\gamma_j L-\sinh\gamma_j L}$$

This leads to the frequency equation $\cos(\gamma_j L)\cosh(\gamma_j L)=1$. This is satisfied by distinct values of $\gamma_j L$, leading to orthogonal mode-shapes. The frequency parameters are calculated by the Newton–Raphson method. The higher-order modeshapes are generated by more accurate methods as given by Gonçalves et al. (2007). The uniform beam natural frequency is $\omega_j=(\gamma_j L)^2\sqrt{\frac{EI_0}{\rho L^4 A_0}}$. The elementary beam modeshapes are orthogonal to each other, and they are input into the Rayleigh–Ritz method. The odd admissible functions are symmetric about the midship, while the even admissible functions are anti-symmetric about the midship.

3.1.2. Rayleigh–Ritz method

Let the vibratory displacement of the non-uniform beam be $z(x,t)=Z(x)\cos\omega t$, where $Z(x)$ is an assumed admissible shape function in space, and ω is the circular frequency. The strain potential energy (U) and the kinetic energy (T) of the beam, as functions of space and time, are expressed as :

$$U(x,t)=\frac{1}{2}EI(x)\left[\frac{d^2Z(x)}{dx^2}\right]^2\bullet\cos^2\omega t\ ;$$
$$T\ (x,t)=\frac{1}{2}m(x)\left[Z(x)\right]^2\bullet\omega^2\sin^2\omega t. \tag{3A,B}$$

The maximum energies are $U_{max}=\frac{1}{2}\int_{x=0}^{x=l}EI(x)\left[\frac{d^2Z(x)}{dx^2}\right]^2dx$;$T_{max}=\omega^2\frac{1}{2}\int_{x=0}^{x=l}m(x)\left[Z(x)\right]^2dx$.

In a conservative system, the maximum potential energy equals the maximum kinetic energy, and thus, the circular frequency can be expressed as

$$\omega^2=\frac{U}{T^*}=\frac{\frac{1}{2}\int_{x=0}^{x=l}EI(x)\left[\frac{d^2Z(x)}{dx^2}\right]^2dx}{\frac{1}{2}\int_{x=0}^{x=l}m(x)\left[Z(x)\right]^2dx} \tag{4}$$

A naturally vibrating system always adjusts itself to its minimum energy configuration. The exact solution would be that of the modeshape $Z(x)$which minimizes the frequency in Eq. (4). To reach the minimum frequency, we assume

$$Z(x)=\sum_{k=1}^{N}a_k\varphi_k(x) \tag{5}$$

The unknown coefficients of Eq. (5), a_k, are calculated by minimizing the frequency with respect to each coefficient. Applying the Ritz method as explained in Thomson (1998) and Timoshenko (1937),

$$\frac{\partial}{\partial a_k}\left\{\frac{\frac{1}{2}\int_{x=0}^{x=l}EI(x)\left[\frac{d^2Z(x)}{dx^2}\right]^2dx}{\frac{1}{2}\int_{x=0}^{x=l}m(x)\left[Z(x)\right]^2dx}\right\}=0 \tag{6}$$

Thus $$\frac{\partial}{\partial a_k}\left\{\frac{1}{2}\int_{x=0}^{x=l}EI(x)\left[\frac{d^2Z(x)}{dx^2}\right]^2dx\right\}-\frac{\left\{\frac{1}{2}\int_{x=0}^{x=l}EI(x)\left[\frac{d^2Z(x)}{dx^2}\right]^2dx\right\}}{\left\{\frac{1}{2}\int_{x=0}^{x=l}m(x)\left[Z(x)\right]^2dx\right\}}\bullet\frac{\partial}{\partial a_k}\left\{\frac{1}{2}\int_{x=0}^{x=l}m(x)\left[Z(x)\right]^2dx\right\}=0$$

or $$\frac{\partial}{\partial a_k}\left\{\frac{1}{2}\int_{x=0}^{x=l}EI(x)\left[\frac{d^2Z(x)}{dx^2}\right]^2dx-\omega^2\bullet\frac{1}{2}\int_{x=0}^{x=l}m(x)\left[Z(x)\right]^2dx\right\}=0. \tag{7}$$

Using the expressions $\lambda=\omega^2$, Generalized stiffness $\alpha_{jk}=\int_0^L EI(x)\varphi_j''(x)\varphi_k''(x)dx$, Generalized mass $\beta_{jk}=\int_0^L m(x)\varphi_j(x)$ the $\varphi_k(x)dx$;
above set of equations (Eq. (7)) reduces to :

$$\sum_{k=1}^{N}a_j\left(\alpha_{jk}-\lambda\beta_{jk}\right)=0 \tag{8}$$

Here, Eq. (8) may be written in the matrix form as follows:

$$\begin{bmatrix}(\alpha_{11}-\lambda\beta_{11})(\alpha_{12}-\lambda\beta_{12})\ldots\ldots\ldots\ldots(\alpha_{1N}-\lambda\beta_{1N})\\(\alpha_{21}-\lambda\beta_{21})(\alpha_{22}-\lambda\beta_{22})\ldots\ldots\ldots\ldots(\alpha_{2N}-\lambda\beta_{2N})\\\cdots\ \cdots\ :\\\cdots\ \cdots\ :\\\cdots\ \cdots\ :\\(\alpha_{N1}-\lambda\beta_{N1})(\alpha_{N2}-\lambda\beta_{N2})\ldots\ldots\ldots\ldots(\alpha_{NN}-\lambda\beta_{NN})\end{bmatrix}\begin{Bmatrix}a_1\\a_2\\:\\:\\a_N\end{Bmatrix}=\begin{Bmatrix}0\\0\\:\\:\\0\end{Bmatrix} \tag{9}$$

For a uniform beam, the matrices of generalized mass β_{jk} and generalized stiffness α_{jk} are diagonal, showing the decoupling of various modes of vibration, each associated with a unique natural frequency, i.e. $\omega_j=\sqrt{\frac{\alpha_{jj}}{\beta_{jj}}}$. For the non-uniform beam, as studied in our paper, the potential energy (PE) associated with a particular uniform beam modeshape influences the kinetic energy (KE) of the other uniform modeshapes, and vice-versa. Thus, the elementary (uniform) beam modeshapes are coupled to each other. The change in the natural frequency from that of the uniform beam is influenced by the mass and stiffness distribution over the length of the beam. A beam which is thicker at the middle and narrower at the ends (approximately similar to a hull girder) is likely to be stiffer. This is due to two reasons:

- Reduced kinetic energy of the lighter ends of the beams, which are the positions of maximum velocity.
- Increased strain potential energy of the stockier middle, where the curvature and flexural rigidity are both maximum.

The determinant of the square matrix of Eq. (9), when equated to zero, gives the frequency equation below.

$$\begin{vmatrix}(\alpha_{11}-\lambda\beta_{11})(\alpha_{12}-\lambda\beta_{12})\ldots\ldots\ldots\ldots(\alpha_{1N}-\lambda\beta_{1N})\\(\alpha_{21}-\lambda\beta_{21})(\alpha_{22}-\lambda\beta_{22})\ldots\ldots\ldots\ldots(\alpha_{2N}-\lambda\beta_{2N})\\\cdots\ \cdots\ :\\\cdots\ \cdots\ :\\\cdots\ \cdots\ :\\(\alpha_{N1}-\lambda\beta_{N1})(\alpha_{N2}-\lambda\beta_{N2})\ldots\ldots\ldots\ldots(\alpha_{NN}-\lambda\beta_{NN})\end{vmatrix}=0. \tag{10}$$

Eq. (10) gives an Nth-order equation in λ, and solving it generates 'N' number of roots : $\lambda_1,\lambda_2,\lambda_3,\ldots\ldots,\lambda_N$, from which we obtain the natural frequencies of the non-uniform beam: $\omega_1,\omega_2,\omega_3,\ldots\ldots,\omega_N$. The non-dimensionalized frequency ω_{jND} is defined as frequency ω_j divided by the factor $\sqrt{\frac{EI_0}{\rho L^4 A_0}}$, whose dimension is sec^{-1}. Here, I_0 is the vertical 2nd moment of inertia of the midship section, and A_0 is the structural (steel) area of the midship. For a uniform beam, it is equal to $(\beta_j L)^2$..

3.1.3. Calculation of the modeshapes of the non-uniform beam (weight-vectors $\boldsymbol{a_{jk}}$)

For $1\le k\le N$, we input λ_k into Eq. (10), in order to re-generate the N × N matrix. Each column corresponds to a weight-vector a_{jk},$1<j,k<N$. This matrix is expected to be diagonally dominant, because $\boldsymbol{\Phi}_k(x)$ has the largest contribution from the $\varphi_k(x)$, and far less from the other elementary modes. The jth row in kth column

gives the weighted contribution of the jth uniform beam modeshape in the kth non-uniform mode. Among the N columns, the particular column that has the largest value for the kth row is chosen as the kth weight-vector, ordered as $a_1 : a_2 : a_3 :\ldots\ldots: a_k :\ldots\ldots: a_N$. Considering the transpose of the weight-vector matrix, the row $a_{j1} : a_{j2} : a_{j3} :\ldots\ldots\ldots: a_{jN}$ is chosen, where 'j' is the index of the chosen row. They are multiplied with the kth elementary modeshape (Eq. (3)) to generate the non-uniform modeshape, which are not orthogonal to other non-uniform modeshapes:

$$\boldsymbol{\Phi}_j(x)=\sum_{k=1}^{N} a_{jk}\varphi_k(x),\ \int_{x=0}^{x=L}\boldsymbol{\Phi}_j(x)\boldsymbol{\Phi}_k(x)dx=\begin{Bmatrix}\sum_{l=1}^{\infty} a_{ll}^2;\ j=k\\ \sum_{l=1}^{\infty} a_{jl}a_{kl};\ j\neq k\end{Bmatrix} \tag{11}$$

3.1.4. Convergence study

A convergence study has been done to ensure that enough uniform beam modes $\varphi_k(x)$ in Eq. (11) are used to satisfactorily derive the non-uniform modeshape $\boldsymbol{\Phi}_j(x)$. Odd non-uniform beam modes require larger contributions from odd uniform beam modeshapes, but there may be traces of the even uniform beam modeshapes, too; and vice-versa. Here, the first few uniform (elementary) modes are included to generate the non-uniform mode. This converges the non-uniform beam frequencies to their final values; and the non-uniform modes to their final shapes.

3.2. Validation with other methods

3.2.1. Finite Element Analysis (FEA)

Since FEA is still the most popular method of finding hull girder natural frequencies, a separate MATLAB code is written to numerically estimate the fundamental and higher order frequencies of the hulls. The energy-based Finite Element Method has been used, obeying the free-free boundary conditions of the non-uniform Euler–Bernoulli beam. The difference between RRM and FEM lies in choosing the shape functions. In RR, the shape function is chosen over the entire beam while in FEM, it is chosen only for one element, as a third order polynomial.

3.2.2. Empirical frequencies

The results are also compared with empirical formulations (Todd, 1961; SR,), which were popular till the 1970's.

a) Todd-type SR94 empirical formulation of 2-noded and 3-noded vertical hull girder vibration are given as $N_{V2}=9.4\times10^4\sqrt{\frac{BD^3}{\Delta L^3}}+19$ cpm, $N_{V3}=13.2\times10^4\sqrt{\frac{BD^3}{\Delta L^3}}+65$ cpm, $N_{Vn}=(n-1)N_{V2}$ cpm.
b) Schilck-type SR94 empirical formulation of 2-noded and 3-noded vertical hull girder vibration are given as: $N_{V2}=27.1\times10^5\sqrt{\frac{I_V}{\Delta L^3}}+14.5$ cpm, $N_{V3}=38.8\times10^5\sqrt{\frac{I_V}{\Delta L^3}}+58.5$ cpm, $N_{Vn}=(n-1)N_{V2}$ cpm.

Here, I_V is the second moment of metal area of cross-section of the midship section.

4. Results

The full analysis under Section 3, including FEA, has been done in MATLAB. This work is limited to the vertical vibration only, since this mode of vibration is well-decoupled from the horizontal-torsional modes of hull girder vibration. For a prismatic beam, $a_{jj}=1, a_{jk}=0$. For the jth natural frequency of the non-uniform beam, the weight a_{jj} is the largest. However, the weight a_{jk}, with $j\neq k$, also has a non-zero contribution to the potential and kinetic energies of the vibrating beam.

Table 2 is the comparative study of the Rayleigh–Ritz method (Section 2), FEA, (Section3.2.1) and empirical estimates (Section3.2.2) of vertical hull girder natural frequencies with respect to published work (Bishop and Price, 1979; Senjanovic et al., 2009; Seng et al., 2012; Zhou and Zhao, 2006; Hirdaris et al., 2003). Apart from the hulls in Section2.1, several other hulls have been considered from available literature, and their mass and stiffness distributions have been image–read through MATLAB. All frequencies are in *rad/sec*.

The 1D equal-size beam element-based FEA, applied on a non-uniform beam, consistently overpredicts the natural frequency by a slight margin, as compared to those by the Rayleigh–Ritz method (column $N=12$). Since the empirical formulations are based on the midship section properties, their frequency estimates are sometimes more and sometimes less than those by the R–R method.

4.1. Frequency convergence study

The convergence of the non-uniform beam natural frequencies, with respect to the increasing number of admissible functions is depicted. Due to the greater discontinuity in the mass distribution in the ballasted case, more admissible functions are required to approximate the energy of the beam, leading to the convergent natural frequencies. It is interesting to note that for the generation of an odd non-uniform frequency, the inclusion of an odd admissible function changes the frequency more than the inclusion of an even admissible function; and vice-versa. The fundamental frequency usually converges using 4–5 admissible functions, till the second decimal place. In ballasted hulls, it may take a few more. The second frequency usually converges with 6–8 admissible functions, while the third requires 9–12 admissible functions.

In Table 2(e), there is a close correspondence with the first two R–R frequencies and the published ones, since both the hulls have been analyzed by the Euler–Bernoulli beam model. The R–R method predicts the fundamental frequency of the destroyer with 0.3% accuracy, with only four (4) admissible functions. The second frequency is within 3% accuracy, with eight(8) admissible functions.

In Table 2(f–g), the literature uses a Timoshenko beam model, while our work has used the Euler–Bernoulli beam model. Still, the first two loaded hull R–R frequencies are comparable to the published ones, with only 5% and 3% discrepancy respectively. The ballasted hull shows more discrepancies due to the distortion of the modeshapes, as compared to the Timoshenko beam.

In Table 2(h–i), the correspondence of the R–R frequencies is milder with the literature, since those works have used the coupled vertical and horizontal torsional vibration of the hull girder, where the system of governing differential equations are statically decoupled but dynamically coupled. Still, the fundamental frequencies are closely corresponded to by the R–R results. In Table 2 (h), the R–R method predicts the fundamental frequency with 5% accuracy, with eight(8) admissible functions.

4.2. Non-uniform modeshape study

Figs. 1–8 show the mass and stiffness distributions of the two hulls studied here, the Destroyer hull from Bishop and Price (1979), and the Bulker hull from Hirdaris et al. (2003), alternately with the first three non-uniform beam modeshapes generated by the Rayleigh–Ritz method. The modeshapes of Tanker hull from

Table 2
Comparative studies of the Rayleigh–Ritz method, FEA, and Empirical estimates of vertical hull girder natural frequencies (rad/s) from published work.

(a) SCIM Containership (Loaded)

Rayleigh–Ritz method									**Empirical**		**FEA**
***N*=4**	***N*=5**	***N*=6**	***N*=7**	***N*=8**	***N*=9**	***N*=10**	***N*=11**	***N*=12**	**Schilck**	**Todd**	
4.75	4.75	4.75	4.74	4.74	–	–	–	–	4.17	6.14	5.18
11.59	11.59	11.57	11.57	–	–	–	–	–	9.93	12.64	12.85
22.62	22.58	22.57	22.53	22.53	22.51	22.50	22.50	–	16.69	24.56	24.52

(b) SCIM Containership (Ballast)

Rayleigh–Ritz method									**Empirical**		**FEA**
***N*=4**	***N*=5**	***N*=6**	***N*=7**	***N*=8**	***N*=9**	***N*=10**	***N*=11**	***N*=12**	**Schilck**	**Todd**	
9.85	9.84	9.84	–	–	–	–	–	–	7.10	10.72	10.19
25.87	25.86	25.80	25.80	25.78	25.78	–	–	–	14.12	19.07	26.56
47.80	46.83	46.76	46.49	46.49	46.45	46.45	46.44	46.44	28.41	42.88	47.86

(c) DS Tanker (Loaded)

Rayleigh–Ritz method									**Empirical**		**FEA**
***N*=4**	***N*=5**	***N*=6**	***N*=7**	***N*=8**	***N*=9**	***N*=10**	***N*=11**	***N*=12**	**Schilck**	**Todd**	
5.86	5.86	–	–	–	–	–	–	–	4.19	6.45	6.16
15.03	15.03	–	–	–	–	–	–	–	9.96	13.08	15.86
29.80	29.78	29.78	29.77	29.77	29.75	29.75	–	–	16.77	25.82	30.90

(d) DS Tanker (Ballast)

Rayleigh–Ritz method									**Empirical**		**FEA**
***N*=4**	***N*=5**	***N*=6**	***N*=7**	***N*=8**	***N*=9**	***N*=10**	***N*=11**	***N*=12**	**Schilck**	**Todd**	
14.74	14.74	14.74	14.73	14.73	–	–	–	–	8.29	13.29	15.64
39.07	39.07	39.03	39.02	39.01	39.00	39.00	–	–	15.82	22.68	41.31
71.79	71.25	71.25	71.05	71.04	70.99	70.97	70.97	70.95	33.16	53.17	75.28

(e) Destroyer, Bishop and Price (1979), Ch.4

Rayleigh–Ritz method									**Lit**	**FEA**
***N*=4**	***N*=5**	***N*=6**	***N*=7**	***N*=8**	***N*=9**	***N*=10**	***N*=11**	***N*=12**		
14.75	14.74	14.74	–	–	–	–	–	–	14.78	16.18
33.03	32.81	32.72	32.70	32.68	32.68	–	–	–	33.78	36.82
62.53	62.09	61.54	61.36	61.34	61.31	61.30	61.30	–	57.25	62.94

(f) Tanker, Bishop and Price (1979), Ch.4 (Loaded)

Rayleigh–Ritz method									**Lit**	**FEA**
***N*=4**	***N*=5**	***N*=6**	***N*=7**	***N*=8**	***N*=9**	***N*=10**	***N*=11**	***N*=12**		
6.33	6.33	–	–	–	–	–	–	–	6.68	6.98
15.35	15.21	15.21	15.21	15.18	15.18	–	–	–	15.69	16.26
32.90	32.54	31.75	31.30	31.21	31.04	30.99	30.97	30.94	29.55	31.41

(g) Tanker, Bishop and Price (1979), Ch.4 (Ballast)

Rayleigh–Ritz method									**Lit**	**FEA**
***N*=4**	***N*=5**	***N*=6**	***N*=7**	***N*=8**	***N*=9**	***N*=10**	***N*=11**	***N*=12**		
3.40	3.40	–	–	–	–	–	–	–	4.00	3.96
8.31	8.31	8.31	8.30	8.30	8.30	8.30	8.30	8.29	10.01	10.31
16.67	16.64	16.63	16.63	16.62	16.61	16.61	16.60	16.60	16.63	18.19

(h) OBO MY Derbyshire (Hirdaris et al., 2003)

Rayleigh–Ritz method									**Empirical**		**Lit**	**FEA**
***N*=4**	***N*=5**	***N*=6**	***N*=7**	***N*=8**	***N*=9**	***N*=10**	***N*=11**	***N*=12**	Schilck	Todd		
4.63	4.63	4.63	4.63	4.62	4.62	–	–	–	3.63	5.75	4.42	4.95
11.06	11.05	11.04	11.03	11.03	11.03	11.02	11.02	–	9.15	12.08	9.25	12.46
22.22	22.21	22.11	22.11	22.11	22.07	22.06	22.04	22.04	14.52	22.99	14.24	23.56

Table 2 (*continued*)

(i) 7800TEU containership, Senjanovic et al. (2009) (loaded)

Rayleigh–Ritz method									Empirical		Lit	FEA
***N*=4**	***N*=5**	***N*=6**	***N*=7**	***N*=8**	***N*=9**	***N*=10**	***N*=11**	***N*=12**	**Schilck**	**Todd**		
3.98	3.97	3.97	–	–	–	–	–	–	3.50	5.49	3.99	4.41
9.86	9.85	9.83	9.81	9.80	9.80	9.80	9.79	9.79	8.97	11.72	8.39	10.86
19.57	19.49	19.41	19.31	19.24	19.20	19.20	19.20	19.20	14.01	21.96	13.18	19.49

(j) 7800TEU containership, Senjanovic et al. (2009) (ballast)

Rayleigh–Ritz method									Empirical		Lit	FEA
***N*=4**	***N*=5**	***N*=6**	***N*=7**	***N*=8**	***N*=9**	***N*=10**	***N*=11**	***N*=12**	**Schilck**	**Todd**		
5.77	5.77	5.77	5.77	5.77	5.76	5.76	–	–	4.31	6.91	5.58	6.22
13.70	13.69	13.67	13.65	13.63	13.63	13.62	13.62	–	10.12	13.72	11.56	14.79
26.47	26.30	26.20	26.11	26.03	25.99	25.99	25.98	25.98	17.24	27.65	17.79	25.83

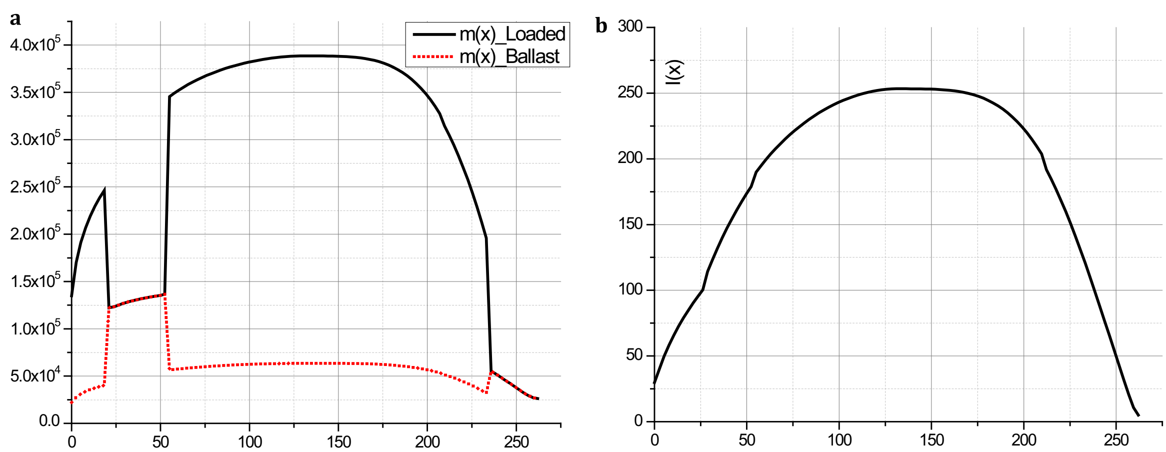

Fig. 1. (**a**,**b**). Mass $m(x)$ and Second moment of Area $I(x)$ distribution of SCIM(Containership), LOA=262 m.

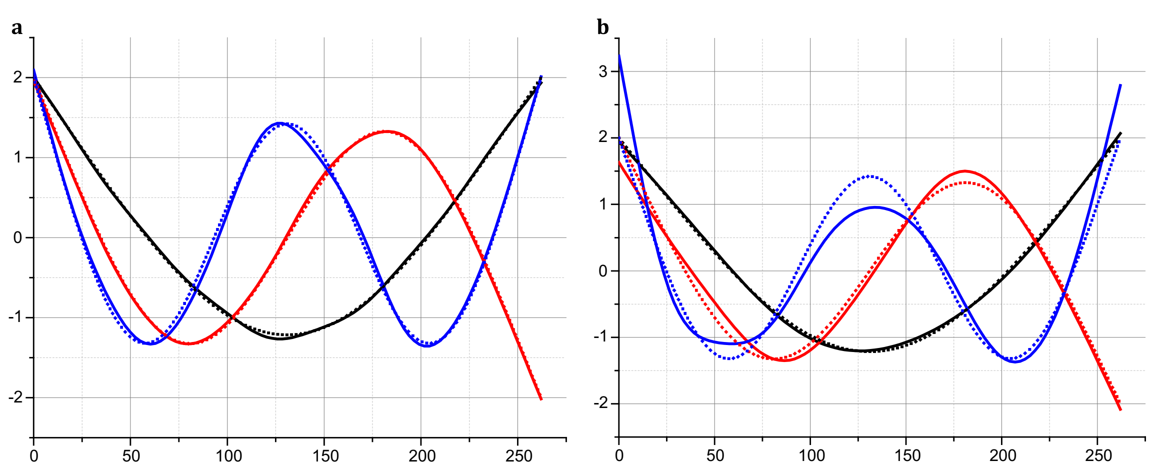

Fig. 2. First three non-uniform beam modeshapes of SCIM (Containership) in the (a) Loaded condition, (b) Ballast condition, by the Rayleigh–Ritz method. Dotted lines show the uniform beam modes.

Bishop and Price have not been compared, since it was studied by the Timoshenko beam theory and our work is limited to the Euler–Bernoulli beam theory. The comparison with the isometric views of the coupled-FEA modeshapes from Senjanovic (2009) is also difficult, and hence, withheld. The non-uniform modeshapes have been achieved through the same number of modeshapes which lead to the frequency convergence, in Section 4.1.

If the non-uniformity of the structure has varied mathematically (linearly, bilinearly, exponentially) along the length, as shown in Serge (1995), Qiusheng et al. (1996), Laura et al. (1996), Zhou and Cheung (2000), Ece et al. (2007), Timoshenko (1937), Thomson et al. (1998) and Auciello (2001), only a few uniform beam modeshape (admissible functions) would be necessary to converge to the non-uniform beam modeshape through the Rayleigh–Ritz

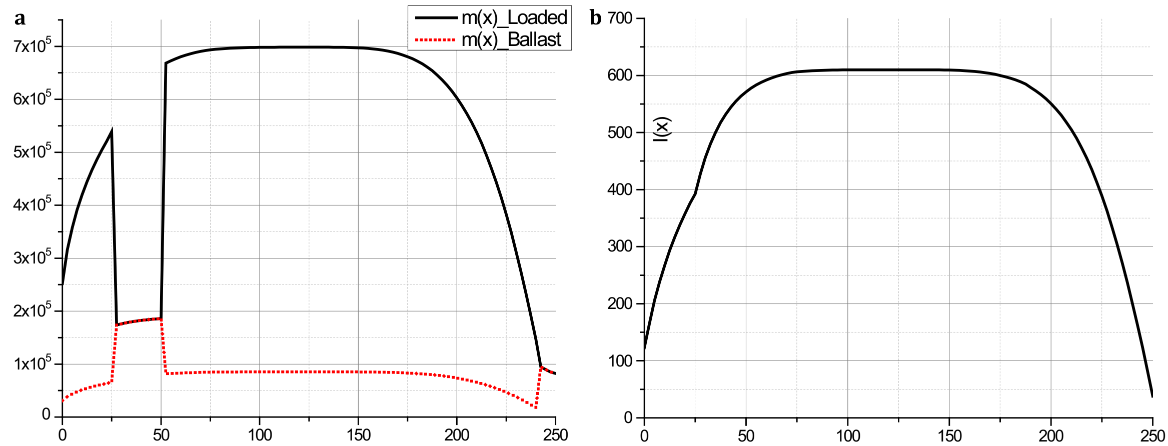

Fig. 3. **(a,b)**. Mass $m(x)$and Second moment of Area $I(x)$ distribution of DS (Tanker), LOA = 250 m.

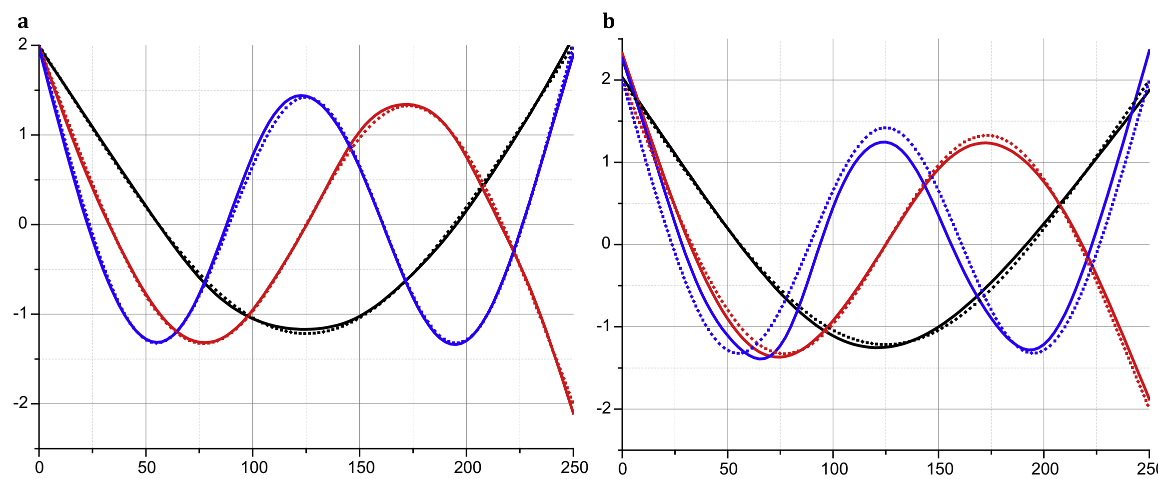

Fig. 4. First three non-uniform beam modeshapes of DS (Tanker) in the (a) Loaded condition, (b) Ballast condition, by the Rayleigh–Ritz method. Dotted lines show the uniform beam modes.

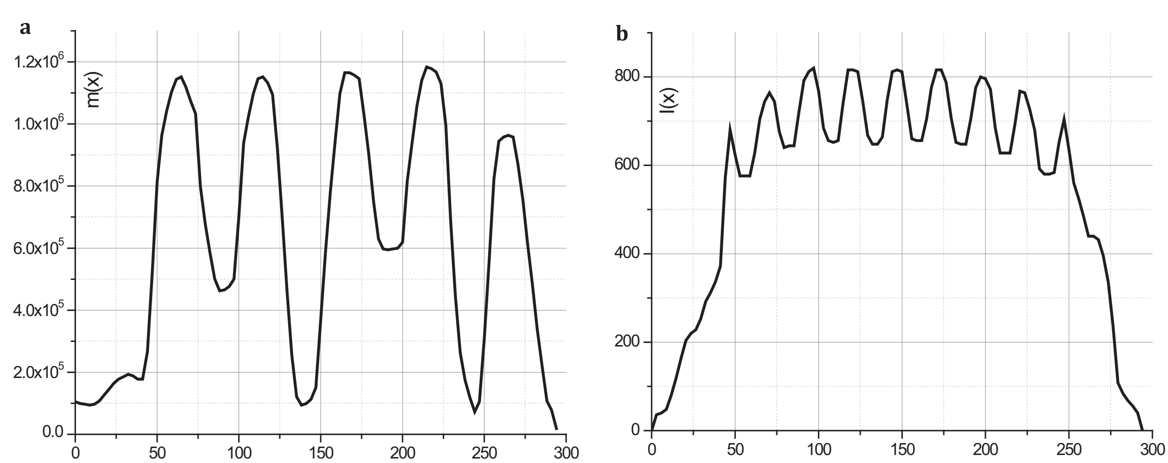

Fig. 5. (a,b). Mass $m(x)$ and Second moment of Area $I(x)$ distribution of OBO MY Derbyshire (Hirdaris et al., 2003), LOA = 294.2 m.

method. The strongly arbitrary variation of the mass and stiffness for a merchant vessel necessitates the inclusion of a few more admissible functions to achieve the frequency and modeshape convergence. Since the mass and stiffness distributions are arbitrary, they distort the non-uniform beam modeshapes from the corresponding uniform ones. Here, 'distortion' would mean : (i) shift in the positions of the nodes and antinodes, (ii) magnification or reduction of the depths of the antinodes, and (iii) more curvature and points of inflexion in the modeshape.

Studying Fig. 2(a,b) and Fig. 4(a,b), it is consistently seen that a *ballasted hull* suffers more distortion of non-uniform modes, since the engine and deck machinery act as beam 'tip masses', causing the ends to deviate away from the admissible functions. A more distorted modeshape has a higher frequency due to more

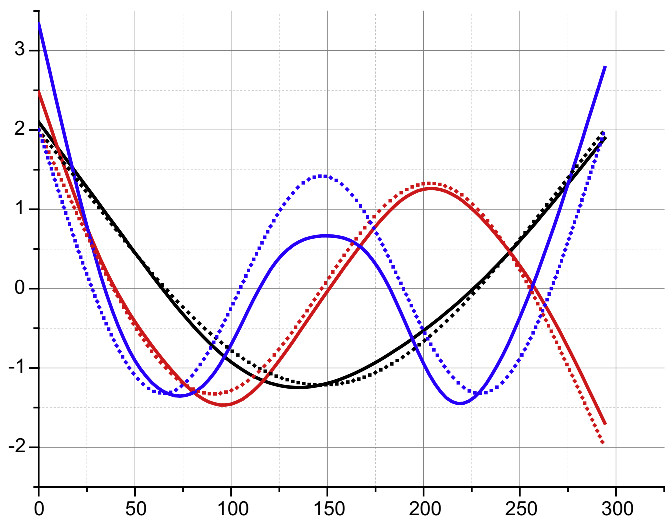

Fig. 6. First three non-uniform beam modeshapes of OBO MY Derbyshire (Hirdaris et al., 2003), by the Rayleigh–Ritz method. Dotted lines show the uniform beam modes.

curvature and potential energy. The ballasted hull has its kinetic energy mostly concentrated at the ends, leading to the lowering of antinodes at the midship, and the increase in antinodes at the ends. A fully loaded modeshape has the modeshape magnitude $\Phi_j(x)<2$, while a ballasted modeshape has a magnitude $\Phi_j(x)<3$. Since modeshapes are spatial shape functions without dimensions, they can be normalized to 1.0 (Bishop and Price, 1979). A *fully loaded ship* has much smoother modeshapes due to a more even mass distribution, leading to an even distribution of kinetic energy. Its non-uniform modeshapes enjoy far less deviations from the corresponding uniform modes; showing the diagonal dominance of the weight vector matrix (Section 3.1.3). Also, presence of large concentrated masses shifts the positions of the nodes/antinodes.

In Fig. 6(b), the R–R modeshapes of the Bulker from Hirdaris et al. (2003) match well with the FEA ones generated in that work (Fig. 6(a)). The odd non-uniform modes are asymmetric about the midships, with the aft end suffering a larger antinode than the fore end. The third mode has a flatter anti-node amidships (compared to the third uniform beam mode), corresponding well with the published modeshape. As seen in Fig. 8(b), the R–R method predicts the odd modeshapes with greater accuracy than the even ones. The first and third modeshapes show good correspondence with the published modeshapes.

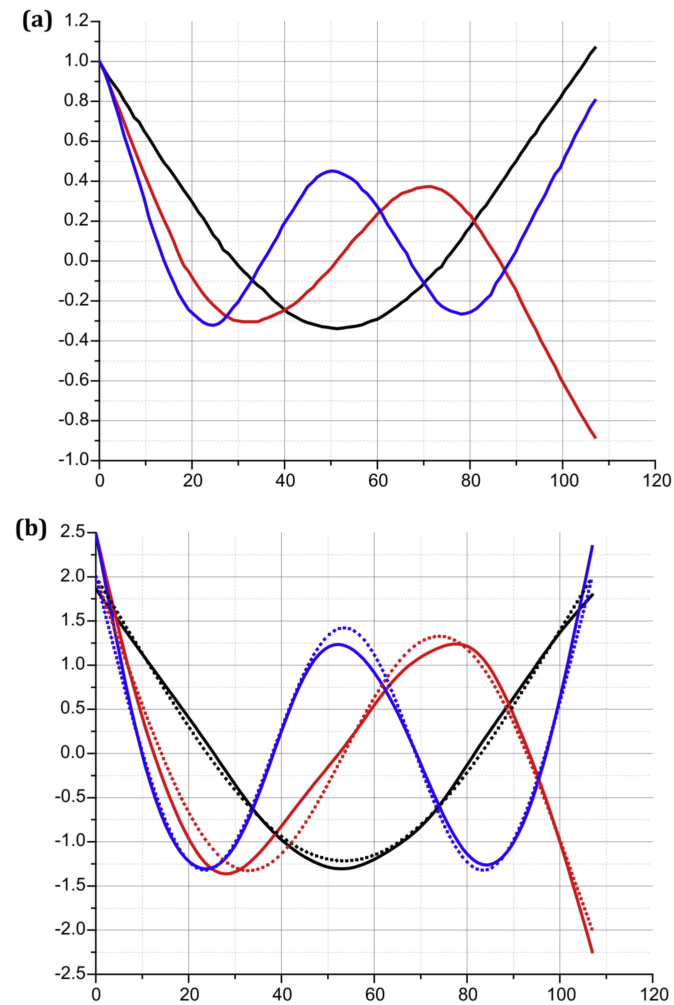

Fig. 8. (a). Data from Bishop and Price (1979), Ch.4, Pages 72–74: First three normalized modeshapes of Destroyer. (b). First three non-uniform beam modeshapes of Destroyer by the Rayleigh–Ritz method, (Bishop and Price, 1979, Ch.4). Dotted lines show the uniform beam modes.

A ballasted hull suffers from larger wave-induced dynamic stress levels. However, a ballasted ship has a higher frequency, further away from the tail-end of the sea-spectrum, and hence much less likely to suffer springing. A loaded ship, with a loaded frequency, is more prone to resonant wave excitations.

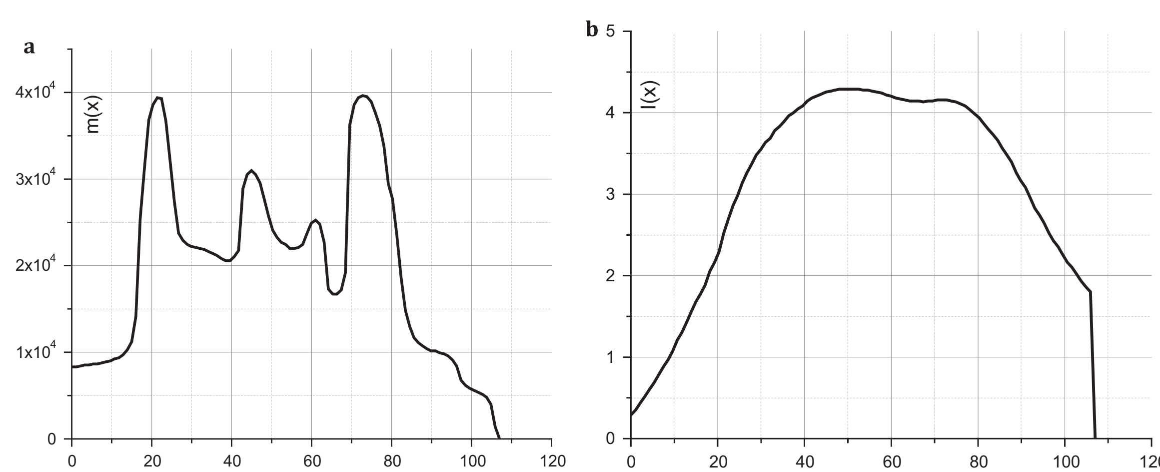

Fig. 7. (a,b). Mass $m(x)$ and Second moment of Area $I(x)$ distribution of Destroyer, LOA = 107 m (Bishop and Price, 1979, Ch.4).

5. Discussion and Conclusions

- *RR with respect to FEA*: The first attempt to bypass the computationally cumbersome and expensive FEA, and use a semi-analytical method, i.e. the energy-based Rayleigh-Ritz method, for hull girder vibration analysis is highlighted. The popularity of FEA in hull girder vibration analysis stems from the fact that it can handle arbitrary geometries. However, it is demonstrated here that Rayleigh-Ritz method can also efficiently handle very arbitrary mass and stiffness distributions, converging to the natural frequencies and modeshapes.
- *First RR-method application in ship hull vibration*: This is the first attempt to study the vibration of non-uniform beams (in general, not ship hull in particular) with very arbitrary longitudinal distribution of mass and flexural rigidity, through energy-based methods. All relevant literature (Serge, 1995; Qiusheng et al., 1996; Laura et al., 1996; Zhou and Cheung, 2000; Ece et al., 2007; Timoshenko, 1937; Thomson et al., 1998; Auciello, 2001; Bishop and Price, 1979) is limited to mathematical variations of sectional properties, and sometimes closed-form non-uniform modeshapes. Attempts to study 'arbitrary' mass distributions have used "lumped masses" on the beam. Attempts to study 'arbitrary' stiffness distributions have used "stepped beam" approximations. Our method uses the most generic case of mass and stiffness distributions, without any mathematical conveniences.
- *Closed-form admissible functions* This method shows reasonable accuracy in prediction of the first few converged dry hull girder vertical vibration frequencies. The use of closed-form admissible functions (uniform beam modeshapes) estimates the energy configuration of the beam faster than FEA. For a 350 m long tanker, 100 elements of the FEA model would mean a 3.5 m long linear straight element, compromising on the energy configuration of the total hull. The closed-form admissible function, as a combination of sinusoidal and exponential functions, gives a closer approximation of the hull shape, slope, and curvature at any x-location.
- *Computational efficiency*: The *hull girder frequencies* predicted by the R–R method act as good initial estimates in the ship design process. This method requires only the first few uniform beam modes (as admissible functions) to generate a converged frequency; compared to FEA which requires much larger number of elements for a smooth hull modeshape, and thus frequency. If 5(five) admissible functions are required to converge the fundamental hull girder frequency, the R–R method operates on a 5×5 matrix built from the energy minimization principle. For the same frequency, a 100-element FEA model works on a 200×200 matrix of the Eigen value problem. This is shown in the convergence study where the solution obtained by RRM converges for much lower number of trial functions (e.g. 4–8) as compared to the number of elements (e.g. 100) in FEM.
- *Future modeshape applications*: The *hull girder modeshapes* predicted by the R–R method can be used as inputs in the hydroelastic analysis of the ship to wave excitation responses, which is done by the efficient normal mode summation method. All flexural forced responses of the hull, in waves at various frequencies and various ship speeds and directions, require the vibration modeshapes of the hull girder for the hydroelastic analysis. Hull modeshapes are also required to study the radiation problem, i.e. wet free vibration of the hull. The modewise added mass distribution of the hull becomes a direct function of each hull modeshape, formulated through the body-boundary condition of 'no penetration'.

References

Aasen, R., Bjørhovde, S. 2010. Early stage weight and cog estimation using parametric formulas and regression on historical data. In: Proceedings of the 69th Annual Conference of Society of Allied Weight Engineers, Inc. Virginia Beach, Virginia, 23–26 May 2010.

Auciello, N.M., 2001. On the transverse vibrations of non-uniform beams with axial loads and elastically restrained ends. Int. J. Mech. Sci. 43, 193 }208.

Bishop, R.E.D., Price, W.G., 1979. Hydroelasticity of Ships. Cambridge University Press, New York.

Camiseiti, C., Macco, A., Polidorou, G., 1980. Investigation of the containership vibration behaviour by finite element method at design stage. Comput. Struct. 12, 395–407.

Ece, Mehmet Cem, Aydogdu, Metin, Taskin, Vedat, 2007. Vibration of a variable cross-section beam. Mech. Res. Commun. 34, 78–84.

Gonçalves, P.J.P., Brennan, M.J., Elliott, S.J., 2007. Numerical evaluation of high-order modes of vibration in uniform Euler–Bernoulli beams. J. Sound Vib. 301.3, 1035–1039.

Hirdaris, S.E., Price, W.G., Temarel, P., 2003. Two- and three-dimensional hydroelastic modeling of a bulker in regular waves. Mar. Struct. 16, 627–658.

Hirdaris, S.E., Bakkers, N., White, N., Temarel, P., 2009. Service factor assessment of a great lakes bulk carrier incorporating the effects of hydroelasticity. SNAME Mar. Technol. J. 46 (2), 116–121.

Kumai, T., 1967. On the estimation of natural frequencies of vertical vibration of ships. J. Soc. Nav. Archit. Jpn. 121, 175–182.

Laura, P.A.A., Gutierrezi, R.H., Rossi, R.E., 1996. Free vibrations of beams of bilinearly varying thickness. Ocean Eng. 23 (I), 1–6.

Datta, N., Thekinen, J.D. "Response spectrum of non-uniform mathematical hull girder springing to deep water random seas with forward speeds", *8th International Conference in Marine Technology,* Malaysia, in October 2012.

Pedersen, P.T., Jensen, J.J., Estimation of hull girder vertical bending moments including non-linear and flexibility effects using closed form expressions. In: Proc. IMechE Part M: J. Engineering for the Maritime Environment. (2009), Vol. 223, pp.377–390.

Qiusheng, Li, Hong, Cao, Guiqing, Li, 1996. Static and dynamic analysis of straight bars with variable cross-section. Comput. Struct. 59 (6) (pp.)1185-l (191).

Sadowski, A.J., 2011. Geometric properties for the design of unusual member cross-sections in bending. Eng. Struct. 33, 1850–1854.

Senjanovic, Ivo, Malenica, S:ime, Tomaševic´, Stipe, 2009. Hydroelasticity of large container ships. Mar. Struct. 22, 287–314.

Serge, Abrate, 1995. Vibration of non-uniform rods and beams. J. Sound Vib. 185 (4), 703–716.

Skaar, K.T., Carlsen, C.A., 1980. Modelling aspects for finite element analysis of ship vibration. Comput. Struct. 12, 409–419.

Sopheak Seng, Ingrid Marie Vincent Andersen, and Jørgen Juncher Jensen, "On the influence of hull girder flexibility on the wave-induced bending moments". In: Proceedings of the sixth International Conference on Hydroelasticity in Marine Technology, Tokyo, JAPAN, 19–21 September 2012.

SR 94 The Research Panel 94 : The shipbuilding Research Association of Japan Report 71–71, No.91.

Thomson, W.T., et al., 1998. Theory of Vibrations with Applications, 5th ed. Pearson Education Inc.

Timoshenko, S., 1937. Vibration Problems in Engineering, 2nd ed. D.Van Nostrand Company, Inc,, New York.

Todd, F.H., 1961. Ship Hull. Vib.

Wan-xic, Zhong, Hui-yu, Xue, Po, Young, 1983. A study of Ship Vibration using finite element method. Appl. Math. Mech. 4 (1), 41–53.

Zhou, D., Cheung, Y.K., 2000. The free vibration of a type of tapered beams. Comput. Methods Appl. Mech. Eng. 188, 203–216.

Zhou, P., Zhao, D., 2006. Analysis of the hull girder vibration by dynamic stiffness matrix method. J. Mar. Sci. Appl. 5 (3), 30–35.

Zhu, S., Wu, M., Moan, T., 2011. Experimental investigation of hull girder vibrations of a flexible backbone model in bending and torsion. Appl. Ocean Res. 33, 252–274.